The Reflection Principle.

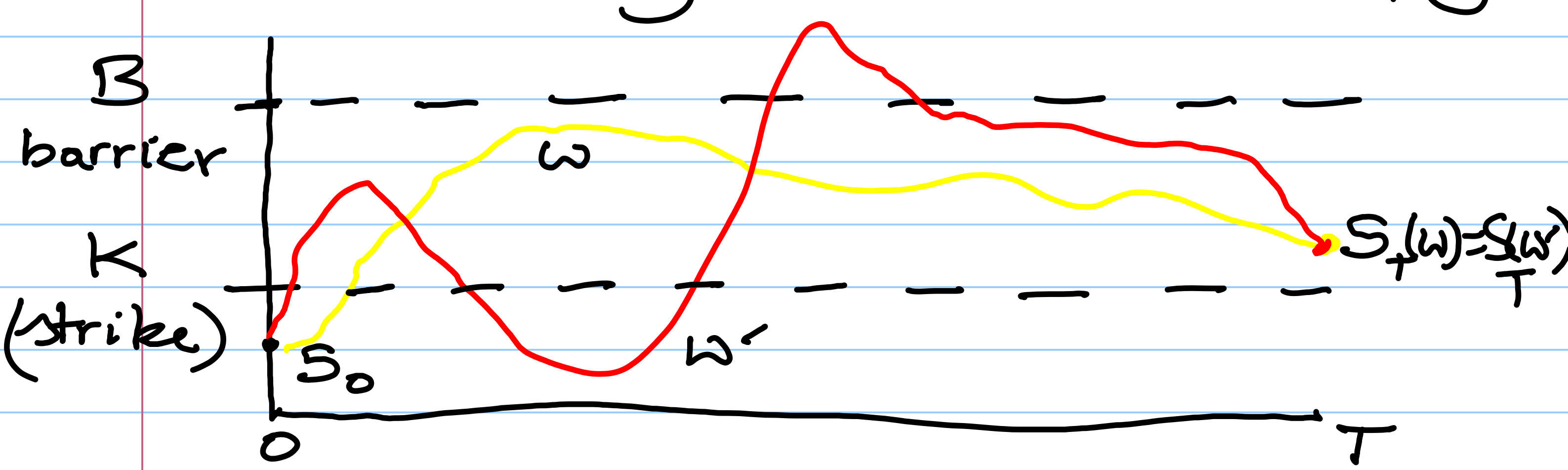
As you may already know, call and put options may be modified by restricting the payoff to a subset of all possible paths;

$$t \longmapsto S_t(\omega) \quad , \quad \omega \in \Omega .$$

One way in which the restriction can occur is to specify that the payoff, $(S_T(\omega)-K)^+$, for example, will occur only if the path

$$t \longmapsto S_t(\omega)$$

does not enter the region, $[B,\infty)$ where B is some "barrier level" agreed at the outset of the contract. This means that paths with the <u>same</u> terminal value, say $S_T(\omega)$, $S_T(\omega')$, with $S_T(\omega) = S_T(\omega')$ but $\omega \neq \omega'$ may have different payoffs



The yellow path gives a payoff of $(S_T(\omega)-K)^+$ but the path ω', in red, gives a payoff of zero! This is because ω' enters the region $[B,\infty)$. So an option with a barrier clause, a barrier option, is path dependant. An option is termed 'knock-out' if entering a region leads to a zero payoff. There are many variants of barrier options, some "knock-in", that is, they have no payoff unless they enter a region. How can we describe this situation in mathematical terms? A first step is to realise that for our knock out barrier option with $S_0 < B$ monitoring the maximum value of S to date, that is,

$$M^S_{0,t} = \max\{S_s(\omega) : s \in [0,t]\}$$

tells us the moment that S enters the region $[B,\infty)$. This because S enters $[B,\infty)$ during $[0,t]$ if and only if

$$M^S_{0,t} \geq B.$$

Exercise: Using the fact that the paths of S are continuous, prove the equivalence stated above.

One can show that $M^S_{0,t}$ is an $\mathcal{F}_t$-adapted random variable and further that

$$\tau(\omega) = \inf\left\{ s : M^S_{0,s} \geq B \right\}$$

is an $(\mathcal{F}_t)$ stopping time: the first time S enters $[B, \infty)$.

A second step is to observe that since, in risk-neutral measure,

$$S_t = S_0 e^{\sigma W_t + (r - \sigma^2/2)t}$$

$$= S_0 e^{\sigma \left(W_t + \left(\frac{r}{\sigma} - \frac{\sigma}{2}\right) t \right)}$$

then

$$S_t(\omega) \geq B \iff W_t + \left(\frac{r}{\sigma} - \frac{\sigma}{2}\right)t \geq \frac{1}{\sigma} \log\left(\frac{B}{S_0}\right)$$

and

$$M^S_t(\omega) \geq B \iff M^X_t \geq \frac{1}{\sigma} \log\left(\frac{B}{S_0}\right)$$

where X is the process $X_t = W_t + \left(\frac{r}{\sigma} - \frac{\sigma}{2}\right)t$.

X is called 'a Brownian Motion with drift', the 'drift' is the deterministic process $\left(\left(\frac{r}{\sigma} - \frac{\sigma}{2}\right)t\right)$. So we have recast the Barrier condition for S in terms of the running maximum of a Brownian motion with drift. For a knock-out barrier option on S we would need to calculate

$$\mathbb{E}^{\mathbb{Q}}\left(\frac{(S_T - K)^+}{e^{rT}} \mathbb{I}_{\{M_T^S < B\}} \right)$$

which we can rewrite in terms of the process X as

$$\mathbb{E}^{\mathbb{Q}}\left(\frac{(S_0 e^{\sigma X_T} - K)}{e^{rT}} \mathbb{I}_{\{X_T \geq \frac{1}{\sigma}\log\left(\frac{K}{S_0}\right)\}} \mathbb{I}_{\{M_T^X < \frac{1}{\sigma}\log\left(\frac{B}{S_0}\right)\}} \right)$$

We could write this in terms of the joint density function of X_T and M_T^X — if we knew that the joint density of X and M^X existed and what it was equal to! —; the calculation would be a double integral, formally,

$$\int_0^{\frac{1}{\sigma}\log\left(\frac{B}{S_0}\right)} \int_{\frac{1}{\sigma}\log\left(\frac{K}{S_0}\right)}^{y} \frac{(e^{\sigma x} - K)}{e^{rT}} f(x,y)\, dx\, dy$$

where $f(x,y)$ is the joint density of (X, M^X).

Finding the joint density of (X, M^X) is not easy! An obvious first case to look at is when X

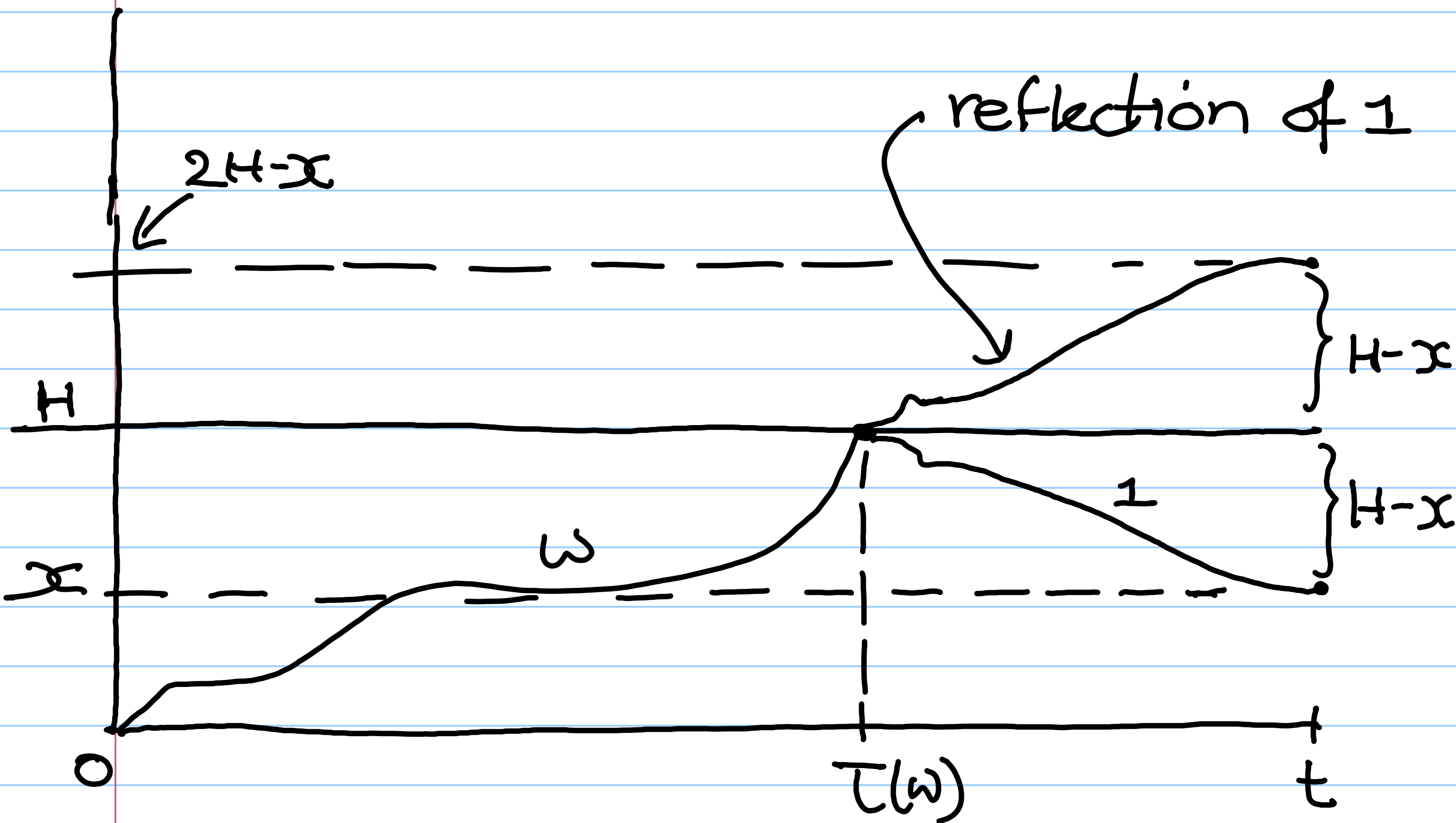
has no drift, so that we just consider a Brownian Motion and its running maximum. It is here that the reflection principle can be employed.

The reflection principle is an heuristic technique which in many cases gives the right answer to questions about Brownian Motion. First of all;

Reborn Brownian Motion:

If we think of Brownian Motion as being the path followed by an infinitesimal point wandering through the real line, we may observe it at time t, it takes the value $W_t(\omega)$. If we ask two questions: first, what is the probability that starting from 0 at time 0 we land up above ($\geq$) the level H at time t? And second, what is the probability that starting from the value $W_t(\omega)$ at time t that we land up above the level $W_t(\omega)+H$ at time $2t$? The answers to these questions are identical. One fancy way of expressing this property is to say that BM is temporaly and spatially homogeneous. That is, for example, the probability that starting from some value, x, at some time t_0 the probability that one lands up above a level $x+H$ after an elapse of time t depends only on H and t and is stochastically independent of x and t_0. We say BM is 'reborn' from every space point at every time.

So now consider the diagram below



Imagine a path, ω, which hits level H for the first time before t and lands up somewhere beneath the level x. Because BM is reborn from the point $(\tau(\omega), H)$ and BM is "symmetric"; for every path that lands up beneath level x there is a reflected path which lands up above level $2H-x$. So, argues the reflection principle, the probability that the running maximum exceeds H over $[0,t]$ and $W_t(\omega)$ is less than x is identical with the probability that the running maximum exceeds H over $[0,t]$ and $W_t(\omega)$ is greater than $2H-x$. In symbols;

$$\mathbb{P}\{M_t^W \geq H, W_t \leq x\} = \mathbb{P}\{M_t^W \geq H, W_t \geq 2H - x\}$$

But observe that $W_t \geq 2H - x \Rightarrow M_t^W \geq H$, since the Brownian paths are continuous. So

$$\mathbb{P}\{M_t^W \geq H, W_t \leq x\} = \mathbb{P}\{W_t \geq 2H - x\}$$

Now, the following union is disjoint

$$\{M_t^W < H, W_t \leq x\} \cup \{M_t^W \geq H, W_t \leq x\}$$

and equal to $\{W_t \leq x\}$. So

$$\mathbb{P}\{W_t \leq x\} = \mathbb{P}\{M_t^W < H, W_t \leq x\} + \mathbb{P}\{M_t^W \geq H, W_t \leq x\}$$
$$= \mathbb{P}\{M_t^W < H, W_t \leq x\} + \mathbb{P}\{W_t \geq 2H - x\}$$

So that,

$$\mathbb{P}\{M_t^W < H, W_t \leq x\} = \mathbb{P}\{W_t \leq x\} - \mathbb{P}\{W_t \geq 2H - x\}$$
$$= N\left(\frac{x}{\sqrt{t}}\right) - N\left(\frac{x - 2H}{\sqrt{t}}\right).$$

Effectively we have calculated the joint distribution of (W_t, M_t^W). From this we can obtain the joint density by effecting $\frac{\partial^2}{\partial H \partial x}$.

All this is very well, but if we try to prosecute the same arguments with $X_t = W_t + \left(\frac{r}{\sigma} - \frac{\sigma}{2}\right)t$ then immediately we run into difficulties. First X does not possess the same kind of symmetry as W:

$$\mathbb{P}\{X_t \geq H\} = \mathbb{P}\left\{W_t \geq H - \left(\frac{r}{\sigma} - \frac{\sigma}{2}\right)t\right\} = N\left(\frac{-H + \left(\frac{r}{\sigma} - \frac{\sigma}{2}\right)t}{\sqrt{t}}\right)$$

$$\mathbb{P}\{X_t \leq -H\} = \mathbb{P}\left\{W_t \leq -H - \left(\frac{r}{\sigma} - \frac{\sigma}{2}\right)t\right\} = N\left(\frac{-H - \left(\frac{r}{\sigma} - \frac{\sigma}{2}\right)t}{\sqrt{t}}\right)$$

A second, more subtle, point is that our argument with the reflection principle assumed, implicitly, that BM is reborn at every stopping time. We have no business assuming this for X!

It turns out that we can use a change of measure technique to calculate

$$\mathbb{E}^{\mathbb{Q}}\left(\frac{(S_0 e^{\sigma X_T} - K)}{e^{rT}} \mathbb{I}_{\left\{X_T > \frac{1}{\sigma}\log\left(\frac{K}{S_0}\right),\ M_T^X < \frac{1}{\sigma}\log\left(\frac{H}{S_0}\right)\right\}}\right)$$

Let $\mathbb{P}$ be the measure

$$\mathbb{P}(E) = \int_E e^{-\nu W_T - \frac{\nu^2}{2} T} \, d\mathbb{Q}$$

with

$$\nu = r/\sigma - \sigma/2 .$$

Under $\mathbb{P}$, $(W_t + \nu t)$ is a Brownian Motion. So, under $\mathbb{P}$,

$$S_t = S_0 e^{\sigma X_t}$$

and $X_t \equiv W_t + \nu t$ is a $\mathbb{P}$-B.M.

Note that

$$\{S_T > K, M_T^S < H\} =$$

$$\{X_T > \tfrac{1}{\sigma} \log\left(\tfrac{K}{S_0}\right), M_T^X < \tfrac{1}{\sigma} \log\left(\tfrac{H}{S_0}\right)\}$$

and the $\mathbb{P}$ probability of this event was calculated in the First part, since $\mathbb{P}$ sees X as a BM.

Notation: Write

$$\hat{K} = \frac{1}{\sigma} \log\left(\frac{K}{S_0}\right), \quad \hat{H} = \frac{1}{\sigma} \log\left(\frac{H}{S_0}\right)$$

then

$$\{S_T > K, M_T^S < H\} = \{X_T > \hat{K}, M_T^X < \hat{H}\}.$$

Let us recall that we want to calculate

$$\mathbb{E}^{\mathbb{Q}}\left((S_T-K) I_{\{S_T>K\}} I_{\{M_T^S<H\}} \right)$$

in terms of X this is

$$\mathbb{E}^{\mathbb{Q}}\left((S_0 e^{\sigma X_T} - K) I_{\{X_T>\hat{K},\, M_T^X<\hat{H}\}} \right)$$

Our problem is that we only have "$\mathbb{P}$-information" in that we have an explicit form for

$$\mathbb{E}^{\mathbb{P}}\{ I_{\{X_T<\hat{K},\, M_T^X<\hat{H}\}} \}.$$

(yes, $X_T<\hat{K}$!)

However, we know that

$$\mathbb{E}^{\mathbb{Q}}(Y) = \mathbb{E}^{\mathbb{P}}\left(Y \frac{d\mathbb{Q}}{d\mathbb{P}}\right)$$

where

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = e^{\nu X_T - \frac{\nu^2}{2}T}$$

(remark, the transition from $\mathbb{Q}$ to $\mathbb{P}$ was achieved by

$$\frac{d\mathbb{P}}{d\mathbb{Q}} = e^{-\nu W_T - \frac{\nu^2}{2}T}$$

while $X_T = W_T + \nu T$ so that

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = e^{\nu(W_T+\nu T)-\frac{\nu^2}{2}T} \quad \text{and}$$

$$\frac{d\mathbb{P}}{d\mathbb{Q}} \cdot \frac{d\mathbb{Q}}{d\mathbb{P}} = e^{-\nu W_T - \frac{\nu^2}{2}T} \cdot e^{\nu W_T + \nu^2 T - \frac{\nu^2}{2}T}$$

$$= e^0 = 1)$$

So

$$\mathbb{E}^{\mathbb{Q}}\left(\frac{(S_0 e^{\sigma X_T} - K)}{e^{rT}} I_{\{X_T > \hat{K},\, M_T^X < \hat{H}\}}\right) =$$

$$\mathbb{E}^{\mathbb{P}}\left(\frac{(S_0 e^{\sigma X_T} - K)}{e^{rT}} I_{\{X_T > \hat{K},\, M_T^X < \hat{H}\}} e^{\nu X_T - \frac{\nu^2}{2}T}\right)$$

and $\mathbb{P}$ regards (X_t) as a BM.

Note that

$$\mathbb{E}^{\mathbb{P}}\left(I_{\{X_T < \hat{K},\, M_T^X < \hat{H}\}}\right) = \quad (\text{yes}, X_T \leq \hat{K})$$

$$N\left(\frac{\hat{K}}{\sqrt{T}}\right) - N\left(\frac{\hat{K} - 2\hat{H}}{\sqrt{T}}\right)$$

From this $\mathbb{P}$ distribution function we can obtain the joint density of X_T and M_T^X under $\mathbb{P}$. However,

$$\mathbb{E}^{\mathbb{Q}}\left(I_{\{X_T \leq \hat{K},\, M_T^X < \hat{H}\}}\right) = \mathbb{E}^{\mathbb{P}}\left(e^{\nu X_T - \frac{\nu^2}{2}T} I_{\{X_T < \hat{K},\, M_T^X < \hat{H}\}}\right)$$

$$= \int_{-\infty}^{\hat{K}} e^{\nu x - \frac{\nu^2}{2}T} \, \mathbb{P}\{X_T = x, \, M_T^X < \hat{H}\} \, dx$$

ϕ is normal density

$$= \int_{-\infty}^{\hat{K}} e^{\nu x - \frac{\nu^2}{2}T} \frac{1}{\sqrt{T}} \left(\phi\left(\frac{x}{\sqrt{T}}\right) - \phi\left(\frac{x - 2\hat{H}}{\sqrt{T}}\right) \right) dx$$

Now

$$\frac{1}{\sqrt{T}} e^{\nu x - \frac{\nu^2}{2}T} \frac{e^{-\frac{1}{2}\left(\frac{x}{\sqrt{T}}\right)^2}}{\sqrt{2\pi T}} = \frac{1}{\sqrt{T}} \phi\left(\frac{x - \nu T}{\sqrt{T}}\right)$$

$\sqrt{T}$

$$-\frac{1}{2}\frac{x^2}{T} + \nu x - \frac{\nu^2}{2}T = -\frac{1}{2}\left(\frac{x^2 - 2T\nu x + \nu^2 T^2}{T}\right)$$

$$= -\frac{1}{2}\left(\frac{x - \nu T}{\sqrt{T}}\right)^2 \quad \lrcorner\!\lrcorner$$

and

$$\frac{1}{\sqrt{T}} e^{\nu x - \frac{\nu^2}{2}T} \, e^{-\frac{1}{2}\left(\frac{x - 2\hat{H}}{\sqrt{T}}\right)^2} =$$

$$= \frac{1}{\sqrt{T}} e^{2\nu\hat{H}} \phi\left(\frac{x - 2\hat{H} - \nu T}{\sqrt{T}}\right) \quad \text{(check this!)}$$

So,

$$\mathbb{E}^{\mathbb{Q}}\left(I_{\{X_T < \hat{K}, \, M_T^X < \hat{H}\}}\right) =$$

$$\int_{-\infty}^{\hat{K}} \frac{1}{\sqrt{T}} \left(\phi\left(\frac{x - \nu T}{\sqrt{T}}\right) - e^{2\nu\hat{H}} \phi\left(\frac{x - 2\hat{H} - \nu T}{\sqrt{T}}\right) \right) dx$$

$$= \mathbb{Q}\{X_T < \hat{K}, \, M_T^X < \hat{H}\} = F_{X_T, M_T^X}(\hat{K}, \hat{H})$$

$$= N\left(\frac{\hat{K}-\nu T}{\sqrt{T}}\right) - e^{2\nu\hat{H}} N\left(\frac{\hat{K}-2\hat{H}-\nu T}{\sqrt{T}}\right)$$

So this is the joint $\mathbb{Q}$ distribution of X_T and M_T^X where $X_T = W_T + \nu T$.

You can proceed as follows. You form,

$$\frac{\partial^2}{\partial\hat{K}\,\partial\hat{H}} F_{X_T, M_T^X}(\hat{K}, \hat{H})$$

and then execute the formal double integral we wrote down earlier. But there is a quicker way. If we form

$$\frac{\partial}{\partial x} F_{X_T, M_T^X}(x, \hat{H})$$

then we can regard

$$\mathbb{Q}\{M_T^X < \hat{H},\ X_T = x\} = \frac{\partial}{\partial x} F_{X_T, M_T^X}(x, \hat{H})\, dx$$

and then

$$\mathbb{E}^{\mathbb{Q}}\left(\frac{(S_0 e^{\sigma X_T} - K)}{e^{rT}} I_{\{M_T^X < \hat{H},\ X_T > \hat{K}\}}\right) =$$

$$\int_{\hat{K}}^{\hat{H}} \frac{(S_0 e^{\sigma x} - K)}{e^{rT}} \left(\frac{1}{\sqrt{T}}\phi\left(\frac{x-\nu T}{\sqrt{T}}\right) - \frac{e^{2\nu\hat{H}}}{\sqrt{T}}\phi\left(\frac{x-2\hat{H}-\nu T}{\sqrt{T}}\right)\right) dx$$

This give a lot of terms we calculate just a few.

First of all

$$e^{2\nu\hat{H}} = e^{2\nu \frac{1}{\sigma}\log\left(\frac{H}{S_0}\right)} = e^{\log\left(\frac{H}{S_0}\right)^{\frac{2\nu}{\sigma}}}$$

$$= \left(\frac{H}{S_0}\right)^{\frac{2}{\sigma}\left(\frac{r}{\sigma}-\frac{\sigma}{2}\right)} = \left(\frac{H}{S_0}\right)^{\frac{2r}{\sigma^2}-1}$$

and we consider the terms involving the strike, K.

$$\int_{\hat{K}}^{\hat{H}} \frac{K}{e^{rT}}\left(\frac{1}{\sqrt{T}}\phi\left(\frac{x-\nu T}{\sqrt{T}}\right) - \left(\frac{H}{S_0}\right)^{\frac{2r}{\sigma^2}-1}\phi\left(\frac{x-2\hat{H}-\nu T}{\sqrt{T}}\right)\right)dx$$

$$= \frac{K}{e^{rT}}\left(N\left(\frac{\hat{H}-\nu T}{\sqrt{T}}\right) - N\left(\frac{\hat{K}-\nu T}{\sqrt{T}}\right)\right) -$$

$$\left(\frac{H}{S_0}\right)^{\frac{2r}{\sigma^2}-1}\frac{K}{e^{rT}}\left(N\left(\frac{-\hat{H}-\nu T}{\sqrt{T}}\right) - N\left(\frac{\hat{K}-2\hat{H}-\nu T}{\sqrt{T}}\right)\right)$$

We can rewrite some of this so that it looks more familiar: Observe, for $A, B \in \mathbb{R}$,

$$N(A) - N(B) = N(A) - 1 + 1 - N(B) = 1 - N(B) - (1 - N(A))$$
$$= N(-B) - N(-A).$$

So $N\left(\frac{\hat{H}-\nu T}{\sqrt{T}}\right) - N\left(\frac{\hat{K}-\nu T}{\sqrt{T}}\right) =$

$$N\left(\frac{\nu T-\hat{K}}{\sqrt{T}}\right)-N\left(\frac{\nu T-\hat{H}}{\sqrt{T}}\right)=$$

$$N\left(\frac{\log\left(\frac{S_0}{K}\right)+\left(r-\sigma^2/2\right)T}{\sigma\sqrt{T}}\right)-N\left(\frac{\log\left(\frac{S_0}{H}\right)+\left(r-\sigma^2/2\right)T}{\sigma\sqrt{T}}\right)$$

this is d_2

We can rewrite $\dfrac{\log\left(\frac{S_0}{H}\right)+\left(r-\sigma^2/2\right)T}{\sigma\sqrt{T}}$.

First of all the exponent in the term $\left(\frac{H}{S_0}\right)^{\frac{2r}{\sigma^2}-1}$ can be written as follows: put

$\lambda=\dfrac{r+\sigma^2/2}{\sigma^2}$, then $\sigma\lambda=\dfrac{r}{\sigma}+\dfrac{\sigma}{2}=\nu+\sigma$

so that $\sigma(\lambda-1)=\nu$. So now,

$$\frac{2r}{\sigma^2}-1=\frac{2}{\sigma}\left(\frac{r}{\sigma}-\frac{\sigma}{2}\right)=\frac{2}{\sigma}\nu$$

$$=2(\lambda-1)$$

and

$$r-\frac{\sigma^2}{2}=\sigma\nu=\sigma^2(\lambda-1)$$

So,

$$\frac{\log\left(\frac{S_0}{H}\right)+\sigma^2(\lambda-1)T}{\sigma\sqrt{T}}=\frac{\log\left(\frac{S_0}{H}\right)+\sigma^2\lambda T}{\sigma\sqrt{T}}-\sigma\sqrt{T}$$

and we write this $= x_1 - \sigma\sqrt{T}$

So these two terms look like,

$$N(d_2) - N(x_1 - \sigma\sqrt{T}).$$

The rest of the calculation follows but please note this is too lengthy to form an exam question. Part of the calculation would be possible for part of an exam question.... but this is speculation.

We consider

$$\frac{K}{e^{rT}}\left(\frac{H}{S_0}\right)^{\frac{2r}{\sigma^2}-1}\left(N\left(\frac{-\hat{H}-\nu T}{\sqrt{T}}\right)-N\left(\frac{\hat{K}-2\hat{H}-\nu T}{\sqrt{T}}\right)\right)$$

$$= \frac{K}{e^{rT}}\left(\frac{H}{S_0}\right)^{2(\lambda-1)}\left(N\left(-\frac{\left(\frac{1}{\sigma}\log\left(\frac{H}{S_0}\right)+\nu T\right)}{\sqrt{T}}\right)\right.$$

$$\left. - N\left(\frac{\frac{1}{\sigma}\log\left(\frac{K}{S_0}\right)-\frac{2}{\sigma}\log\left(\frac{H}{S_0}\right)-\nu T}{\sqrt{T}}\right)\right)$$

using $N(A)-N(B) = N(-B)-N(-A)$, so

$$= \frac{K}{e^{rT}}\left(\frac{H}{S_0}\right)^{2(\lambda-1)}\left(N\left(\frac{\frac{2}{\sigma}\log\left(\frac{H}{S_0}\right)-\frac{1}{\sigma}\log\left(\frac{K}{S_0}\right)+\nu T}{\sqrt{T}}\right) - \right.$$

$$\left. N\left(\frac{\frac{1}{\sigma}\log\left(\frac{H}{S_0}\right)+\nu T}{\sqrt{T}}\right)\right)$$

$$= \frac{K}{e^{rT}}\left(\frac{H}{S_0}\right)^{2(\lambda-1)}\left(N\left(\frac{\log\left(\frac{H^2}{KS_0}\right)+\sigma\nu T}{\sigma\sqrt{T}}\right) - N\left(\frac{\log\left(\frac{H}{S_0}\right)+\sigma\nu T}{\sigma\sqrt{T}}\right)\right)$$

$$= \frac{K}{e^{rT}}\left(\frac{H}{S_0}\right)^{2(\lambda-1)} N\left(\frac{\log\left(\frac{H^2}{KS_0}\right)+\sigma^2(\lambda-1)T}{\sigma\sqrt{T}}\right) - N\left(\frac{\log\left(\frac{H}{S_0}\right)+\sigma^2(\lambda-1)T}{\sigma\sqrt{T}}\right)\Bigg)$$

Write $\dfrac{\log\left(H^2/KS_0\right)+\sigma^2\lambda T}{\sigma\sqrt{T}} = y$ and

$$\frac{\log\left(\frac{H}{S_0}\right)+\sigma^2\lambda T}{\sigma\sqrt{T}} = y_1$$

then these terms can be written as

$$\frac{K}{e^{rT}}\cdot\left(\frac{H}{S_0}\right)^{2(\lambda-1)}\left(N\left(y-\sigma\sqrt{T}\right) - N\left(y_1-\sigma\sqrt{T}\right)\right)$$

Bearing in mind that these terms involving the strike are subtracted from those involving $S_0 e^{\sigma x_T}$ we can write this part of our calculation as

$$\frac{K}{e^{rT}}\left(N\left(x_1-\sigma\sqrt{T}\right) - N(d_2)\right) + \frac{K}{e^{rT}}\left(\frac{H}{S_0}\right)^{2(\lambda-1)}\left(N\left(y_1-\sigma\sqrt{T}\right) - N\left(y-\sigma\sqrt{T}\right)\right).$$

We look at the other terms

arising from,

$$\int_{\hat{K}}^{\hat{H}} \frac{S_0 e^{\sigma x}}{e^{rT}\sqrt{T}}\left(\phi\left(\frac{x-\nu T}{\sqrt{T}}\right) - e^{2\nu\hat{H}}\phi\left(\frac{x-2\hat{H}-\nu T}{\sqrt{T}}\right)\right)dx$$

Now $2\nu\hat{H} = \frac{2\nu}{\sigma}\log\left(\frac{H}{S_0}\right)$ so $e^{2\nu\hat{H}} = \left(\frac{H}{S_0}\right)^{2(\lambda-1)}$

First of all we consider

$$\int_{\hat{K}}^{\hat{H}} \frac{e^{\sigma x}}{\sqrt{T}}\phi\left(\frac{x-\nu T}{\sqrt{T}}\right)dx$$

we 'complete the square' in the exponential term

$$\sigma x - \frac{1}{2}\left(\frac{x-\nu T}{\sqrt{T}}\right)^2 = \sigma x - \frac{1}{2T}(x-\nu T)^2$$

$$= \frac{-1}{2T}\left((x-\nu T)^2 - 2T\sigma x\right)$$

$$= \frac{-1}{2T}\left(x^2 - 2\nu T x + \nu^2 T^2 - 2T\sigma x\right)$$

$$= \frac{-1}{2T}\left(x^2 - 2x(\nu+\sigma)T + \nu^2 T^2\right)$$

$$= -\frac{1}{2T}\left(\left(x-(\nu+\sigma)T\right)^2 - (\nu+\sigma)^2 T^2 + \nu^2 T^2\right)$$

$$= \frac{-1}{2T}\left(\left(x-(\nu+\sigma)T\right)^2 - 2\nu\sigma T^2 - \sigma^2 T^2\right)$$

$$= -\frac{1}{2}\left(\frac{x-(\nu+\sigma)T}{\sqrt{T}}\right)^2 + \nu\sigma T + \frac{\sigma^2 T}{2}, \text{ and}$$

$$\left(\frac{r}{\sigma} - \frac{\sigma}{2}\right)\sigma T + \frac{\sigma^2 T}{2} = rT \quad . \quad \text{So,}$$

$$\int_{\hat{K}}^{\hat{H}} \frac{S_0}{e^{rT}} \frac{e^{\sigma x}}{\sqrt{T}} \phi\left(\frac{x-\nu T}{\sqrt{T}}\right) dx \quad \text{is}$$

$$\int_{\hat{K}}^{\hat{H}} \frac{S_0}{e^{rT}} \frac{1}{\sqrt{T}} \phi\left(\frac{x-(\nu+\sigma)T}{\sqrt{T}}\right) e^{rT} dx \quad \text{which is}$$

$$S_0 \int_{\hat{K}}^{\hat{H}} \frac{1}{\sqrt{T}} \phi\left(\frac{x-(\nu+\sigma)T}{\sqrt{T}}\right) dx = S_0 \left[N\left(\frac{x-(\nu+\sigma)T}{\sqrt{T}}\right)\right]_{\hat{K}}^{\hat{H}}$$

$$= S_0\left(N\left(\frac{\hat{H}-(\nu+\sigma)T}{\sqrt{T}}\right) - N\left(\frac{\hat{K}-(\nu+\sigma)T}{\sqrt{T}}\right)\right)$$

$$= S_0\left(N\left(\frac{\log\left(\frac{H}{S_0}\right) - (r+\sigma^2/2)T}{\sigma\sqrt{T}}\right) - N\left(\frac{\log\left(\frac{K}{S_0}\right) - (r+\sigma^2/2)T}{\sigma\sqrt{T}}\right)\right)$$

using $N(A) - N(B) = N(-B) - N(-A)$ this is

$$= S_0\left(N\left(\frac{\log\left(\frac{S_0}{K}\right) + (r+\sigma^2/2)T}{\sigma\sqrt{T}}\right) - N\left(\frac{\log\left(\frac{S_0}{H}\right) + (r+\sigma^2/2)T}{\sigma\sqrt{T}}\right)\right)$$

$= S_0(N(d_1) - N(x_1))$, x_1 defined above and 'd_1' the 'usual thing'.

Now we look at, $\int_{\hat{K}}^{\hat{H}} e^{\sigma x} \frac{S_0}{e^{rT}} \frac{1}{\sqrt{T}} e^{2\nu\hat{H}} \phi\left(\frac{x-2\hat{H}-\nu T}{\sqrt{T}}\right)$

(I left off the dx .... no room). First of all,

$$e^{2\nu\hat{H}} = e^{\frac{2\nu}{\sigma}\log\left(\frac{H}{S_0}\right)} = \left(\frac{H}{S_0}\right)^{\frac{2\nu}{\sigma}} .$$

We complete the square in the exponential terms:

$$\sigma x - \frac{1}{2T}(x - 2\hat{H} - \nu T)^2 =$$

$$-\frac{1}{2T}\left(x^2 - 2x(2\hat{H}+\nu T) + (2\hat{H}+\nu T)^2 - 2\sigma x T\right) =$$

$$-\frac{1}{2T}\left(x^2 - 2x(2\hat{H}+\nu T + \sigma T) + (2\hat{H}+\nu T)^2\right)$$

add in $(2\hat{H} + (\nu+\sigma)T)^2$ and take it away this gives a quadratic term

$$(x - (2\hat{H} + (\nu+\sigma)T))^2$$

and a residual

$$(2\hat{H}+\nu T)^2 - (2\hat{H} + (\nu+\sigma)T)^2 =$$

$$\cancel{4\hat{H}^2} + \cancel{4\hat{H}\nu T} + \nu^2 T^2 - \cancel{4\hat{H}^2} - \cancel{4\hat{H}(\nu+\sigma)T} - (\nu+\sigma)^2 T^2$$

$$= \cancel{\nu^2 T^2} - 4\hat{H}\sigma T - \cancel{\nu^2 T^2} - 2\nu\sigma T^2 - \sigma^2 T^2$$

$$= -4\hat{H}\sigma T - 2\nu\sigma T^2 - \sigma^2 T$$

and,

$$-\frac{1}{2T}\left(-4\hat{H}\sigma T - 2\nu\sigma T^2 - \sigma^2 T^2\right) = 2\hat{H}\sigma + \nu\sigma T + \frac{\sigma^2 T}{2}$$

So these terms produce, first, $e^{2\log\left(\frac{H}{S_0}\right)}$

$= \left(\frac{H}{S_0}\right)^4$ which combines with $\left(\frac{H}{S_0}\right)^{\frac{2\nu}{\sigma}}$;

$$\frac{2\nu}{\sigma} + 2 = 2(\lambda - 1) + 2 = 2\lambda$$

while $\left(\nu\sigma + \frac{\sigma^2}{2}\right)T = \left(r - \frac{\sigma^2}{2} + \frac{\sigma^2}{2}\right)T$

$= rT$ and the terms e^{-rT} and e^{rT} cancel. We integrate,

$$S_0\left(\frac{H}{S_0}\right)^{2\lambda} \int_{\hat{K}}^{\hat{H}} \frac{1}{\sqrt{T}}\, \phi\left(\frac{x - (2\hat{H} + (\nu+\sigma)T)}{\sqrt{T}}\right) dx$$

$$= S_0\left(\frac{H}{S_0}\right)^{2\lambda}\left[N\left(\frac{-\hat{H} - (\nu+\sigma)T}{\sqrt{T}}\right) - N\left(\frac{\hat{K} - 2\hat{H} - (\nu+\sigma)T}{\sqrt{T}}\right)\right]$$

$$= S_0\left(\frac{H}{S_0}\right)^{2\lambda}\left[N\left(\frac{\log\left(\frac{S_0}{H}\right) - (r+\sigma^2/2)T}{\sigma\sqrt{T}}\right) - N\left(\frac{\log\left(\frac{KS_0}{H^2}\right) - (r+\sigma^2/2)T}{\sigma\sqrt{T}}\right)\right]$$

Since $r + \frac{\sigma^2}{2} = \sigma^2\lambda$ we can rewrite these terms as

$$S_0\left(\frac{H}{S_0}\right)^{2\lambda}\left(N(-y_1) - N(-y)\right)$$

Of course we must not forget that all of this is preceded by "$-$". So our formula for the up and out barrier option is

$$S_0\left(N(d_1)-N(x_1) + \left(\frac{H}{S_0}\right)^{2\lambda}\left(N(-y) - N(-y_1)\right)\right) +$$

$$Ke^{-rT}\left(-N(d_2)+N(x_1-\sigma\sqrt{T}) - \left(\frac{H}{S_0}\right)^{2(\lambda-1)}\left(N(-y+\sigma\sqrt{T})-N(-y_1+\sigma\sqrt{T})\right)\right)$$